

ON THE STABILISATION HEIGHT OF FIBRE SURFACES IN S^3

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ABSTRACT. The stabilisation height of a fibre surface in the 3-sphere is the minimal number of Hopf plumbing operations needed to attain a stable fibre surface from the initial surface. We show that families of fibre surfaces related by iterated Stallings twists have unbounded stabilisation height.

1. INTRODUCTION

A famous theorem by Giroux and Goodman states that every fibre surface in S^3 admits a common stabilisation with the standard disc [6]. More precisely, every fibre surface $\Sigma \subset S^3$ can be obtained from the embedded standard disc by a finite sequence of Hopf plumbing operations, followed by a finite number of Hopf deplumbing operations. The necessity of the deplumbing operation was known for a long time, thanks to examples of Melvin and Morton [8]. In this note, we show that the number of deplumbing operations is unbounded, even for fibre surfaces of genus zero. We define the stabilisation height of a fibre surface $\Sigma \subset S^3$ as

$$h(\Sigma) = \min_S \{b_1(S) - b_1(\Sigma)\},$$

where b_1 denotes the first Betti number and the minimum is taken over all common stabilisations $S \subset S^3$ of Σ and the disc. The stabilisation height measures the number of Hopf plumbing operations needed to pass from Σ to a minimal common stabilisation S with the disc, since Hopf plumbing increases the first Betti number of a surface by one.

Theorem 1. *Let $\Sigma_n \subset S^3$ be a family of fibre surfaces of the same topological type Σ whose monodromies $\varphi_n : \Sigma \rightarrow \Sigma$ differ by a power of a Dehn twist along an essential simple closed curve $c \subset \Sigma$:*

$$\varphi_n = \varphi_0 T_c^n.$$

Then

$$\lim_{|n| \rightarrow \infty} h(\Sigma_n) = +\infty.$$

An easy way of producing such families is by applying iterated Stallings twists to a fixed fibre surface. A detailed example of this sort, as well as a definition of a Stallings twist, are presented in the next section. We do not know whether all families of monodromies related by powers of a Dehn twist arise from an iterated Stallings twist. In any case, not all fibre surfaces support Stallings twists. Indeed, a Stallings twist requires the existence of an essential, unknotted, 0-framed simple closed curve in the fibre surface. In particular, quasipositive surfaces do not admit Stallings twists.

Question. *Does there exist a family of quasipositive fibre surfaces $\Sigma_n \subset S^3$ of the same topological type with*

$$\lim_{n \rightarrow \infty} h(\Sigma_n) = +\infty?$$

The existence of quasipositive fibre surfaces $\Sigma \subset S^3$ (or equivalently, pages of open books supporting the standard contact structure on S^3) with $h(\Sigma) \geq 1$ was recently established by Baker et al. [1], Wand [11] and, for the genus zero case, Etnyre-Li [4].

The proof of Theorem 1 makes use of the stable commutator length of the monodromy. The key observation is that a Hopf plumbing operation corresponds to composing the monodromy with a Dehn twist, which cannot change the commutator length too much. On the other hand, a large power of a Dehn twist has a large commutator length, thanks to a result by Endo-Kotschick and Korkmaz [3, 7]. A detailed proof is given in Section 3. As a preparation, we discuss the Dehn twist length of the monodromy and present a simple family of fibre surfaces of genus zero illustrating Theorem 1.

2. THE TWIST LENGTH OF THE MONODROMY

The monodromy of an iterated Hopf plumbing involving n Hopf bands is a product of n Dehn twists. Hopf plumbing decreases the Euler characteristic of a surface by one. A surface of genus g with one boundary component has Euler characteristic $1 - 2g$. Therefore, if a fibred knot can be obtained from the unknot by Hopf plumbing, its monodromy can be written as a product of $2g$ Dehn twists. The following proposition (taken from the second author's PhD thesis [9]) shows that the number $2g$ is in fact minimal.

Proposition 1. *Let K be a fibred knot of genus g with monodromy φ . Then any representation of φ as a product of Dehn twists involves at least $2g$ distinct factors.*

The proof of Proposition 1 has three main ingredients. Firstly, the Alexander polynomial $\Delta_K(t) \in \mathbb{Z}[t]$ of a fibred knot K with fibre surface Σ equals the characteristic polynomial of the homological action $\varphi_* : H_1(\Sigma, \mathbb{R}) \rightarrow H_1(\Sigma, \mathbb{R})$ of the monodromy φ , that is,

$$\Delta_K(t) = \det(\text{id} - \varphi_*).$$

Secondly, if K is a knot, $\Delta_K(1) = 1$. This does not hold for links of more than one component: in fact, $\Delta_L(1) = 0$ if L is a link with at least two components. Thirdly, the homological action of a Dehn twist T about a simple closed curve γ in Σ can be described as follows. Let α be any simple closed curve in Σ and denote by $a = [\alpha]$ and $c = [\gamma]$ the classes of α and γ in $H_1(\Sigma, \mathbb{R})$. Then we have (see [5])

$$T_*(a) = a + i(a, c)c,$$

where $i(\cdot, \cdot)$ denotes the intersection pairing.

Proof of Proposition 1. Suppose to the contrary that φ could be written as a product of Dehn twists with $n < 2g$ distinct factors T_1, \dots, T_n , where $T_i = T_{c_i}$ denotes a Dehn twist about a simple closed curve c_i in Σ . Let

$$V = \langle [c_1], \dots, [c_n] \rangle \subset H_1(\Sigma, \mathbb{R})$$

be the subspace of $H_1(\Sigma, \mathbb{R})$ spanned by the classes of the c_i . Consider the orthogonal complement of V in $H_1(\Sigma, \mathbb{R})$ with respect to the intersection form,

$$V^\perp = \{x \in H_1(\Sigma, \mathbb{R}) \mid i(x, y) = 0 \quad \forall y \in V\}.$$

Since $\dim V \leq n < 2g = \dim H_1(\Sigma, \mathbb{R})$ and i is non-degenerate, we have $\dim V^\perp \geq 2g - n > 0$, so there exists a non-zero vector $v \in V^\perp$. We claim that v is an eigenvector of φ_* for the eigenvalue 1. Indeed, we may write v as a finite linear combination $v = \lambda_1 a_1 + \dots + \lambda_r a_r$, with $\lambda_k \in \mathbb{R}$ and where $a_1, \dots, a_r \in H_1(\Sigma, \mathbb{R})$ can be represented by simple closed curves $\alpha_1, \dots, \alpha_r$ in Σ . For every fixed $k \in \{1, \dots, r\}$, we have

$$\begin{aligned} T_k(v) &= \sum_{j=1}^r \lambda_j T_k(a_j) = \sum_{j=1}^r \lambda_j (a_j + i(a_j, c_k)c_k) \\ &= \sum_{j=1}^r \lambda_j a_j + i\left(\sum_{j=1}^r \lambda_j a_j, c_k\right)c_k = v + i(v, c_k)c_k = v + 0 \cdot c_k, \end{aligned}$$

hence $\varphi_*(v) = v$, as claimed. But then, $1 = \Delta_K(1) = \chi_{\varphi_*}(1) = 0$, a contradiction. \square

Definition 1. Let Σ be an abstract surface with boundary and let f be a mapping class of Σ fixing the boundary pointwise. We define the *twist length* $t(f)$ to be the minimal number of factors in a representation of f as a product of (positive and negative) Dehn twists:

$$t(f) = \min\{k \in \mathbb{N} \mid f = T_1 \cdots T_k, \quad T_i \text{ Dehn twist}\}$$

Remark. The twist length $t(f)$ is the word length of f as an element of the mapping class group $\text{Mod}(\Sigma, \partial\Sigma)$ with respect to the generating set given by all Dehn twists along essential simple closed curves in Σ .

Remark. In terms of twist length, Proposition 1 implies $t(\varphi) \geq 2g$ whenever φ is the monodromy of a fibred knot of genus g . The examples in Subsection 2.2 below show that the twist length can in fact be arbitrarily large for monodromies of fibred links of fixed genus.

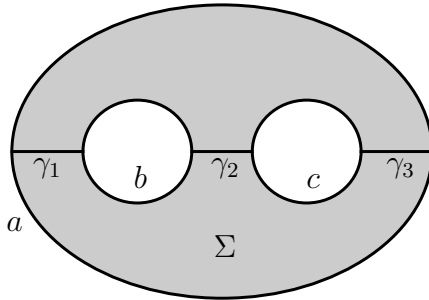


FIGURE 1. A pair of pants, its three boundary curves a, b, c and the three non-separating arcs $\gamma_1, \gamma_2, \gamma_3$.

2.1. Stallings twist. Let $\Sigma \subset S^3$ be a fibre surface with monodromy φ and let $c \subset \Sigma$ be a 0-framed simple closed curve which is unknotted in S^3 . A *Stallings twist* along c consists of a ± 1 Dehn surgery along c^+ , a curve obtained by pushing c slightly off Σ in the positive normal direction. Stallings realised that this operation turns Σ into another fibre surface Σ' of the same topological type, whose monodromy is the map φ composed with T_c^\pm (compare [10]).

By Giroux and Goodman's theorem [6], the effect of a Stallings twist on Σ can equally be achieved by plumbing and deplumbing Hopf bands. In particular cases (including the examples in Subsection 2.2 below), this can be seen explicitly, as remarked by Melvin and Morton (see Lemma, Figure 7 and Figure 8 in [8]).

2.2. Example. Let Σ be an oriented pair of pants and denote a, b, c its three boundary curves. For every given integer n , consider the mapping class

$$\varphi_n = T_a T_b^{-1} T_c^n,$$

where T_a, T_b, T_c are right-handed Dehn twists about (curves parallel to) a, b, c . Note that these Dehn twists commute since we can choose the corresponding twist curves to be disjoint. The following holds:

- (1) $t(\varphi_n) = n + 2$,
- (2) there is a fibre surface $\Sigma_n \subset S^3$ whose monodromy is φ_n ,
- (3) $\lim_{|n| \rightarrow \infty} h(\Sigma_n) = +\infty$.

The first statement follows from the fact that a, b, c represent the only isotopy classes of simple closed curves in Σ , whence the mapping class group of Σ is the free abelian group generated by T_a, T_b, T_c ; compare [5, Section 3.6.4].

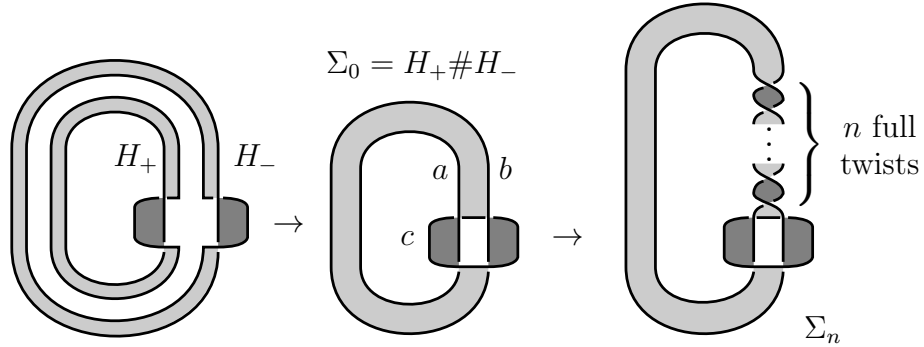


FIGURE 2. Stallings twist along a curve c on a connected sum of a positive and a negative Hopf band.

For the second statement, let H_+, H_- be a positive Hopf band with core curve α and a negative Hopf band with core curve β , respectively. The connected sum of H_+ and H_- is a fibre surface $\Sigma_0 \subset S^3$ homeomorphic to Σ . Up to a permutation of a, b, c , we may assume that the homeomorphism sends (α, β) to (a, b) , so that the monodromy of Σ_0 is given by $\varphi_0 = T_a T_b^{-1}$. The curve $c \subset \Sigma$ corresponds to an unknotted 0-framed curve in Σ_0 , along which we can perform an n -fold Stallings twist. This turns Σ_0 into the fibre surface Σ_n depicted in Figure 2 on the right, whose monodromy is φ_n .

The third statement is an application of Theorem 1. It implies in particular that Σ_n cannot be an iterated Hopf plumbing on the standard disc for large n . This can also be seen directly as follows. Suppose

that Σ_n were an iterated plumbing of Hopf bands. Undoing the last Hopf plumbing amounts to cutting the surface Σ_n along a properly embedded non-separating arc, and the resulting surface is itself a fibre surface. However, Σ_n is homeomorphic to a pair of pants, which contains exactly three non-separating proper arcs up to isotopy (compare [5, Proposition 2.2]). Let $\gamma_1, \gamma_2, \gamma_3$ be the arcs connecting the boundary curves a, b , respectively b, c and a, c , as shown in Figure 1. Cutting Σ_n along any of the arcs $\gamma_1, \gamma_2, \gamma_3$ results in an unknotted annulus with $0, n+1, n-1$ full twists, respectively. Since the only fibred annuli are the positive and the negative Hopf band, we conclude that Σ_n cannot be a Hopf plumbing for $|n| \geq 3$.

3. THE STABLE COMMUTATOR LENGTH OF THE MONODROMY

The goal of this section is to prove Theorem 1 by estimating the effect of an iterated Dehn twist on the stable commutator length of the monodromy. Let G be a perfect group, i.e. a group in which every element is a finite product of commutators. The commutator length $\text{cl}(g)$ of an element $g \in G$ is defined as the minimal number of commutators required in a factorisation of g into commutators. The *stable commutator length* is defined as the following limit, whose existence follows from the sub-additivity of the function $n \mapsto \text{cl}(g^n)$:

$$\text{scl}(g) = \lim_{n \rightarrow \infty} \frac{1}{n} \text{cl}(g^n).$$

The interested reader is referred to Calegari's monograph [2] for more background on the stable commutator length.

We will be concerned with the stable commutator length on mapping class groups of orientable surfaces of genus greater than two, a famous family of perfect groups. The main ingredient in our proof of Theorem 1 is an estimate for the stable commutator length of a Dehn twist T_c along an essential simple closed curve $c \subset \Sigma$, where Σ is a closed orientable surface of genus $g \geq 3$:

$$\text{scl}(T_c) \geq \frac{1}{18g-6}.$$

This was first proved by Endo-Kotschick for Dehn twists along separating curves [3], then by Korkmaz for all essential curves [7].

Proof of Theorem 1. Let $\Sigma_n \subset S^3$ be a family of fibre surfaces of the same topological type Σ whose monodromies differ by a power of a Dehn twist along an essential simple closed curve $c \subset \Sigma$:

$$\varphi_n = \varphi_0 T_c^n.$$

Let $S_n \subset S^3$ be a common stabilisation of the standard disc and the fibre surface $\Sigma_n \subset S^3$. By plumbing at most six additional Hopf bands to Σ_n , we can make sure that

- (i) the genus of S_n is at least three,
- (ii) the complementary component(s) of the curve $c \subset S_n$ have genus at least one.

Let $k \in \mathbb{N}$ be the number of Hopf plumbings needed to get from Σ_n to S_n . The corresponding monodromy $\overline{\varphi}_n : S_n \rightarrow S_n$ can be written as a product of b_1 Dehn twists, where $b_1 = b_1(S_n)$ is the first Betti number of the surface S_n . As a consequence, the stable commutator length of $\overline{\varphi}_n$ is bounded from above by a constant $C(b_1)$ that depends only on b_1 :

$$\text{scl}(\overline{\varphi}_n) \leq C(b_1). \quad (1)$$

Indeed, there is an upper bound on the commutator length of Dehn twists for a surface with finite first Betti number, since there are only finitely many conjugacy classes of Dehn twists in the mapping class group of a fixed surface.

The remainder of the proof consists in showing

$$\lim_{|n| \rightarrow \infty} \text{scl}(\overline{\varphi}_n) = +\infty,$$

provided that $b_1(S_n)$ is bounded while $|n|$ tends to infinity. In order to estimate the stable commutator length $\text{scl}(\overline{\varphi}_n)$, we cap off the surface S_n by discs (abstractly, not in S^3) and extend the monodromy $\overline{\varphi}_n$ by the identity on these discs. We thus obtain a diffeomorphism

$$\widetilde{\varphi}_n : \widetilde{S}_n \rightarrow \widetilde{S}_n$$

of a closed surface \widetilde{S}_n of genus at least three whose stable commutator length satisfies

$$\text{scl}(\overline{\varphi}_n) \geq \text{scl}(\widetilde{\varphi}_n).$$

By construction, the map $\widetilde{\varphi}_n$ can be expressed as

$$\widetilde{\varphi}_n = T_k T_{k-1} \dots T_1 \varphi_0 T_c^n,$$

where T_1, T_2, \dots, T_k are Dehn twists along the core curves of the k Hopf bands plumbed to Σ_n . An elementary calculation (compare Section 2.7.4 on free products in [2]) shows

$$\text{scl}(gh) \geq \text{scl}(g) + \text{scl}(h) - 1,$$

for arbitrary elements g, h of a perfect group. This yields

$$\begin{aligned} \text{scl}(\widetilde{\varphi}_n) &\geq \text{scl}(T_k T_{k-1} \dots T_1 \varphi_0) + \text{scl}(T_c^n) - 1 \\ &\geq \text{scl}(T_k T_{k-1} \dots T_1) + \text{scl}(\varphi_0) + \text{scl}(T_c^n) - 2 \\ &\geq \dots \\ &\geq \sum_{i=1}^k \text{scl}(T_i) + \text{scl}(\varphi_0) + \text{scl}(T_c^n) - (k+1). \end{aligned}$$

At last, we apply Korkmaz' result (Theorem 2.1 in [7]) to the curve $c \subset \widetilde{S}_n$, which is non-trivial, since its complementary regions have strictly positive genus (compare (ii)): $\text{scl}(T_c) > 0$. This implies

$$\lim_{|n| \rightarrow \infty} \text{scl}(T_c^n) = \lim_{|n| \rightarrow \infty} |n| \text{scl}(T_c) = +\infty,$$

hence

$$\lim_{|n| \rightarrow \infty} \text{scl}(\overline{\varphi}_n) = +\infty,$$

provided that k (equivalently $b_1(S_n)$) is bounded while $|n|$ tends to infinity. Here again we use the fact that there is an upper bound on the commutator length of Dehn twists for a surface with finite first Betti number. This is in contradiction with (1). We thus conclude

$$\lim_{|n| \rightarrow \infty} b_1(S_n) = +\infty,$$

which is precisely the statement of Theorem 1: the stabilisation height of S_n tends to infinity as $|n|$ does. \square

REFERENCES

- [1] K. Baker, J. B. Etnyre, J. Van Horn-Morris: *Cabling, contact structures and mapping class monoids*, J. Differential Geom. **90** (2012), no. 1, 1–80.
- [2] D. Calegari: *scl*, MSJ Memoirs, 20. Mathematical Society of Japan, Tokyo, 2009.
- [3] H. Endo, D. Kotschick: *Bounded cohomology and non-uniform perfection of mapping class groups*, Invent. Math. **144** (2001), no. 1, 169–175.
- [4] J. Etnyre, Y. Li: *The arc complex and contact geometry: nondestabilizable planar open book decompositions of the tight contact 3-sphere*, Int. Math. Res. Not. IMRN 2015, no. 5, 1401–1420.
- [5] B. Farb, D. Margalit: *A primer on mapping class groups*, Princeton University Press, 2012.
- [6] É. Giroux, N. Goodman: *On the stable equivalence of open books in three-manifolds*, Geom. Topol. **10** (2006), 97–114.
- [7] M. Korkmaz: *Stable commutator length of a Dehn twist*, Michigan Math. J. **52** (2004), no. 1, 23–31.
- [8] P. M. Melvin, H. R. Morton: *Fibred knots of genus 2 formed by plumbing Hopf bands*, J. London Math. Soc. (2) **34** (1986), no. 1, 159–168.

- [9] F. Misev: *On the plumbing structure of fibre surfaces*, PhD thesis, University of Bern, 2016.
- [10] J. R. Stallings: *Constructions of fibred knots and links*, Algebraic and geometric topology, Proc. Sympos. Pure Math. XXXII (1978), Part 2, 55–60, Amer. Math. Soc., Providence, R.I.
- [11] A. Wand: *Factorizations of diffeomorphisms of compact surfaces with boundary*, Geom. Topol. **19** (2015), no. 5, 2407–2464.

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